Smital properties of Fubini products of σ-ideals

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$\mathrm{Bor}(X)$ – the family of Borel subsets of X.
$\mathcal{I}, \mathcal{J}$ – σ-ideals on Polish spaces.
$\mathcal{M}, \mathcal{N}$ – the families of meager and null sets respectively.
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Definition

We say that a σ-ideal $\mathcal{I}$

- is invariant if $x+A\in\mathcal{I}$ and $-A\in\mathcal{I}$ for every $x\in X$ and $A\in\mathcal{I}$;
- has a Borel base if

$$(\forall A\in\mathcal{I})(\exists B\in \mathrm{Bor}(X)\cap\mathcal{I})(A\subseteq B).$$

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Definition

We call a set A

- $\mathcal{I}$-positive if $A\notin\mathcal{I}$;
- $\mathcal{I}$-residual if $A^c\in\mathcal{I}$.

Let $\mathcal{A}$ be a σ-algebra on X that is a base for $\mathcal{I}$.

Definition

We say that a pair $(\mathcal{A}, \mathcal{I})$ has

- the Smital Property if

$$(\forall D \subseteq X)(\forall B \in \mathcal{A} \setminus \mathcal{I})(D \text{ is dense } \rightarrow (B + D)^c \in \mathcal{I});$$

- the Weaker Smital Property if

$$(\exists D \subseteq X, D \text{ dense and ctbl })(\forall B \in \mathcal{A} \setminus \mathcal{I})((B + D)^c \in \mathcal{I});$$

- the Very Weak Smital Property if

$$(\forall B \in \mathcal{A} \setminus \mathcal{I})(\exists D \subseteq X, D \text{ dense and ctbl })((B + D)^c \in \mathcal{I}).$$

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$(\mathrm{Bor}, \mathcal{M})$ and $(\mathrm{Bor}, \mathcal{N})$ have all of them while $\mathcal{M} \cap \mathcal{N}$ and smaller σ-ideals have none.

Let $A \subseteq X \times Y$, $x \in X$ and $y \in Y$. Denote

$$A_x = \{y \in Y : (x, y) \in A\} \quad \text{(vertical slice of A in x)}$$
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Let $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$ be (σ)-algebras and denote by $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ the (σ)-algebra generated by rectangles of the form $A \times B$, $A \in \mathcal{A}$, $B \in \mathcal{B}$.

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Definition

Let $\mathcal{I} \subseteq P(X)$ and $\mathcal{J} \subseteq P(Y)$ be (σ-)ideals. Then

$$\mathcal{I} \otimes \mathcal{J} = \{A \subseteq X \times Y : (\exists C \in \mathcal{C})(A \subseteq C \ \wedge \ \{x \in X : C_x \notin \mathcal{J}\} \in \mathcal{I})\}$$

is the Fubini product of $\mathcal{I}$ and $\mathcal{J}$.

Definition

Let $\mathcal{F} \subseteq P(X)$, $\mathcal{G} \subseteq P(Y)$, $\mathcal{H} \subseteq P(X \times Y)$ be families of sets. Then we say that $\mathcal{G}$ is $\mathcal{H}$-on-$\mathcal{F}$ if for each set $H \in \mathcal{H}$

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Main cases:

- $\mathcal{G} = \mathcal{J} \subseteq P(Y)$ is a σ-ideal;
- $\mathcal{F} \in \{Bor(X), \sigma(\mathrm{Bor}(X) \cup \mathcal{I})\}$;
- $\mathcal{H} \in \{Bor(X \times Y), \sigma(Bor(X \times Y) \cup \mathcal{I} \otimes \mathcal{J})\}$;

where $\sigma(\mathrm{Bor}(X) \cup \mathcal{I})$ is the σ-algebra of $\mathcal{I}$- measurable sets.

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where $\sigma(\text{Bor}(X) \cup \mathcal{I})$ is the σ-algebra of $\mathcal{I}$- measurable sets. Examples: $\mathcal{M}$ and $\mathcal{N}$ are Borel-on-Borel and measurable-on-measurable.

Bartoszewicz A., Filipczak M., Natkaniec T., On Smital Properties, Topology and its Applications, vol. 158 (2011), 2066-2075.

Let $\mathcal{I} \subseteq P(X), \mathcal{J} \subseteq P(Y)$ be (σ-)ideals and $\mathcal{A} \subseteq P(X), \mathcal{B} \subseteq P(Y)$ be (σ-)algebras.

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Let $\mathcal{I} \subseteq P(X), \mathcal{J} \subseteq P(Y)$ be (σ-)ideals and $\mathcal{A} \subseteq P(X), \mathcal{B} \subseteq P(Y)$ be (σ-)algebras. Erroneous claim: each pair $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ has the following property.

Definition

We say that a pair $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ have the Positive Rectangle Property if for every $\mathcal{I} \otimes \mathcal{J}$-positive set $C \in \mathcal{A} \otimes \mathcal{B}$ there are $A \in \mathcal{A} \backslash \mathcal{I}$ and $B \in \mathcal{B} \backslash \mathcal{J}$ such that $A \times B \subseteq C \cup K$ for some $K \in \mathcal{I} \otimes \mathcal{J}$.

Proposition

The pair $(\mathrm{Bor}(\mathbb{R}^2), [\mathbb{R}]^{\leq\omega} \otimes [\mathbb{R}]^{\leq\omega})$ *does not have the Positive Rectangle Property.*

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$(\mathrm{Bor}(\mathbb{R}^2), \mathcal{M})$ has the Positive Rectangle Property. What about $\mathcal{N}$?

Theorem (Erdös, Oxtoby)

There is a set $G \subseteq \mathbb{R}^2$ such that $G \cap (A \times B)$ and $G^c \cap (A \times B)$ have a positive measure for each $A, B \subseteq \mathbb{R}$ of positive measure.

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Proof.

Take $G = \{(x, y) \in \mathbb{R}^2 : x - y \in F\}$, where F and F^c have a non-null intersection with every nonempty open set. □

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Corollary

$(\mathrm{Bor}(\mathbb{R}^2), \mathcal{N})$ *does not have the Positive Rectangle Property.*

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Let $\mathcal{K} \subseteq P(X \times Y)$ be an ideal and $(\mathcal{A} \otimes \mathcal{B})[\mathcal{K}]$ be an algebra generated by $(\mathcal{A} \otimes \mathcal{B}) \cup \mathcal{K}$

Corollary

Each pair $((\mathcal{A} \otimes \mathcal{B})[\mathcal{K}], \mathcal{K})$ of algebra-ideal has the Positive Rectangle Property.

Theorem

Let $(\mathrm{Bor}(X), \mathcal{I})$ *and* $(\mathrm{Bor}(Y), \mathcal{J})$ *possess the Weaker Smital Property and assume one of the following properties*

1. $\mathcal{J}$ *is Borel-on-Borel;*
2. $\mathcal{J}$ *measurable-on-measurable;*
3. $(\mathrm{Bor}(X \times Y), \mathcal{I} \otimes \mathcal{J})$ *has the Positive Rectangle Property.*

Then $(\mathrm{Bor}(X \times Y), \mathcal{I} \otimes \mathcal{J})$ *also has the Weaker Smital Property.*

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- Take $B \in \mathrm{Bor}(X \times Y) \setminus \mathcal{I} \otimes \mathcal{J}$.

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- Let D_1 and D_2 witness the Weaker Smital Property for $\mathcal{I}$ and $\mathcal{J}$ respectively.

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- Let D_1 and D_2 witness the Weaker Smital Property for $\mathcal{I}$ and $\mathcal{J}$ respectively.
- $(D_1 \times D_2) + B \supseteq \bigcup_{d_1 \in D_1} \bigcup_{x \in \widetilde{B}} (\{d_1 + x\} \times (D_2 + B_x))$ is $\mathcal{I} \otimes \mathcal{J}$-residual.

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Corollary

Let $n \in \omega$ and $\mathcal{I}_k \in \{\mathcal{M}, \mathcal{N}\}$ for any $k \leq n$. Then $(\mathrm{Bor}, \mathcal{I}_0 \otimes \mathcal{I}_1 \otimes \ldots \otimes \mathcal{I}_n)$ has the Weaker Smital Property.

For a given σ-ideal $\mathcal{I}$ denote $\mathcal{I}' = \{A^c : A \in \mathcal{I}\}$.

Theorem (A. Bartoszewicz, M. Filipczak, T. Natkaniec)

If $(\mathcal{A},\mathcal{I})$ has the Smital property and $\mathcal{B} = \mathcal{J} \cup \mathcal{J}'$ then $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ also has it.

Definition

We say that a pair $(\mathcal{A}\otimes\mathcal{B}, \mathcal{I}\otimes\mathcal{J})$ has the Tall Rectangle Hull Property (TRHP) if for every set $C\in\mathcal{A}\otimes\mathcal{B}$

$$(\exists\widetilde{C}\in\mathcal{A}, I\in\mathcal{I}, J\in\mathcal{J})((\widetilde{C}\setminus I)\times(Y\setminus J)\subseteq C\subseteq(\widetilde{C}\times Y)\cup(I\times Y)\cup(X\times J)).$$

Analogously we define the Wide Rectangle Hull Property (WRHP).

Lemma

The family of sets possessing TRHP is closed under countable unions and complements. The same is true for the family of sets possessing WRHP.

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Proposition

Let $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$. Then

1. *if $\mathcal{A} = \mathcal{I} \cup \mathcal{I}'$ then $(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has WRHP.*
2. *if $\mathcal{B} = \mathcal{J} \cup \mathcal{J}'$ then $(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has TRHP.*

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The family of sets possessing TRHP is closed under countable unions and complements. The same is true for the family of sets possessing WRHP.

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2. *if $\mathcal{B} = \mathcal{J} \cup \mathcal{J}'$ then $(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has TRHP.*

Theorem

Let $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ and assume that

1. *$(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has TRHP and $(\mathcal{A}, \mathcal{I})$ has the Smital Property, or*
2. *$(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has WRHP and $(\mathcal{B}, \mathcal{J})$ has the Smital Property.*

Then $(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has the Smital Property.

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The following are equivalent:

1. $(\mathrm{Bor}(X), \mathcal{I})$ *has the Very Weak Smital Property;*
2. $\mathcal{I}$ *is maximal among invariant proper* σ*-ideals with a Borel base.*

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Are $\mathcal{M}$ *and* $\mathcal{N}$ *the only maximal invariant* σ*-ideals with Borel bases on* $\mathbb{R}$*?*

Theorem

There are $\mathfrak{c}$ *many maximal invariant* σ*-ideals with Borel bases on* 2^ω*.*

Thank you for your attention!

Michalski M., Rałowski R., Żeberski Sz., Ideals with Smital properties, arXiv:2102.03287v2